RESEARCH ARTICLE OPEN ACCESS

A Coupled Fixed Point Theorem for Geraghty Contractions in Partially Ordered Metric Spaces

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Abstract
In this paper we establish results on the existence and uniqueness of coupled fixed points of Geraghty contraction on a partially ordered set with a metric, with the continuity of the altering distance function dropped. Our results are improvements over the results of GVR Babu and P.Subhashini [3].
Key words: Geraghty contraction, altering distance function, fixed point, mixed monotone property, partially ordered set.
Mathematical subject classification (2010): 47H10, 54H25

I. Introduction

Delbosco [8] and Skof [23] introduced the concept of an altering distance function, which alters the distance between two points in a metric space. This technique is made famous by Khan, Swaleh and Sessa [15]. Afterwards many researchers [2, 6, 9, 10, 15, 22] applied this concept to obtain the existence of fixed points. Also, G.V.R. Babu and P. Subhashini [3] applied this concept to obtain coupled fixed point via Geraghty contraction.
Throughout this paper $\mathbb{R}^+$ denotes the interval $[0,\infty)$.

II. Preliminaries

Notation: Let $\Phi = \{\varphi: \mathbb{R}^+ \to \mathbb{R}^+ / \varphi$ is non decreasing and $\varphi(t) = 0 \Leftrightarrow t = 0\}$

Definition 2.1: [23] $\boldsymbol{\varphi} \in \Phi$ is called an altering distance function if φ is continuous.

Definition 2.2: [14] Let $(X, \leq)$ be partially ordered set and $F: X \times X \to X$ be a map. We say that f has the mixed monotone property if $F(x,y)$ is non-decreasing in x and is non-increasing in$y$.
i.e. $x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$ and $y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2)$.

Definition 2.3: [12] Let $(X, \leq)$ be a partially ordered set and $F: X \times X \to X$ be a map. For any$x, y \in X$. A point $(x,y) \in X \times X$ is called a coupled fixed point of F if $x = F(x,y)$ and $y = F(y,x)$.
Define the partial order $\leq_1$ on $X \times X$ as follows:
$(x,y) \leq_1 (u,v)\ if\ x \leq u\ \ and\ y \geq v\ \forall\, x, y, u, v \in X.$
We say that (x,y) and (u,v) are comparable, if either $(x,y) \leq_1 (u,v)$ or $(u,v) \leq_1 (x,y)$
We denote the class of all altering distance functions $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ by Φ. We use the following notation as mentioned in [11] to define Geraghty contraction.
$S = \{\beta: \mathbb{R}^+ \to [0,1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$

Definition 2.4: [11] Let (X,d) be a metric space. A self map $f: X \to X$ is said to be a Geraghty contraction if $\exists\, \beta \in S$ such that $d\big(f(x), f(y)\big) \leq \beta\big(d(x,y)\big)\, d(x,y)\ \forall\, x, y \in X$

Remark 2.5: It is trivial to see that every contraction map is Geraghty contraction. The following example (GVR Babu and P Subhashini [3]) shows that a Geraghty contraction need not be a contraction.

Example 2.6: [3] Let $X = \mathbb{R}^+$ with usual metric. We define $f: X \to X$ by $f(x) = \frac{x}{1+x}$ $\forall\, x \in X$ and $\beta: \mathbb{R}^+ \to [0,1)$ by $\beta(t) = \begin{cases} \frac{2}{2+t} & if \;\; t > 0 \\ 0 & if \;\; t = 0 \end{cases}$

Then clearly $\beta \in S$. We observe that f is a Geraghty contraction. For,

$$d\big(f(x), f(y)\big) = \left|\frac{x}{1+x} - \frac{y}{1+y}\right| = \frac{|x-y|}{(1+x)(1+y)} \le \frac{2|x-y|}{2+|x-y|} = \beta\big(d(x,y)\big)\, d(x,y)\ \forall x, y \in X$$

But f is not a contraction.

The following theorem is proved in [11].

Theorem 2.7 (Geraghty, [11]): Let $f: X \to X$ be a self map of a complete metric space X. If $\exists\, \beta \in S$ such that $d\big(f(x), f(y)\big) \le \beta\big(d(x,y)\big) d(x,y)\ \forall\, x, y \in X$, then for any choice of initial point x_0, the iteration $x_n = f(x_{n-1})$ for $n = 1,2,3, \ldots$ converges to the unique fixed point z of f in X.

In 2010 Amini-Harandi and Emami [1] extended Theorem 2.7 to complete metric spaces with partial order.

Theorem 2.8 [1]: Let $(X, \le)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f: X \to X$ be an increasing mapping and $\exists\, \beta \in S$ such that $d\big(f(x), f(y)\big) \le \beta\big(d(x,y)\big)\, d(x,y)\ \forall\, x, y \in X$ with $x \ge y$.

Assume that either (i) f is continuous or (ii) X is such that if $\{x_n\}$ an increasing sequence $x_n \to x$ in X, then $x_n \le x\ \forall\, n$.

Besides suppose thatif for each $x, y \in X, \exists\, z \in X$ which is comparable to x and y. Then f has a unique fixed point.

Guo and Lakshmikantham [13] introduced the mixed monotone operators. In 2006, Gnana Bhaskar and Lakshmikantham [12] established the existence of coupled fixed points for mixed monotone operators in metric spaces with partial order. For more literature on the existence of coupled fixed points of different contraction conditions in partially ordered metric spaces, we refer [4, 7, 14, 16, 18, 19, 20, 21].

Definition 2.9: [12] Let X be a non-empty set and $F: X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is said to be coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

The following theorem is due to Gnana Bhaskar and Lakshmikantham [12].

Theorem 2.10: [12] Let $(X, \le)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0,1)$ such that

$$d(F(x,y), F(u,v)) \le \frac{k}{2}[d(x,u) + d(y,v)] \qquad \ldots\ldots\ (1)$$

for all $x, y, u, v \in X$ with $x \ge u, y \le v$.

Also suppose that either (i) F is continuous or (ii) X has the following properties:

(a) If $\{x_n\}$ is a non-decreasing sequence in X with $x_n \to x$, then $x_n \le x\ \forall\, n \in \mathbb{Z}^+$

(b) If $\{y_n\}$ is a non- increasing sequence in X with $y_n \to y$, then $y_n \ge y\ \forall\, n \in \mathbb{Z}^+$

If $\exists\, x_0, y_0 \in X$ such that $x_0 \le F(x_0, y_0)$ and $y_0 \ge F(x_0, y_0)$, then $\exists\, x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$

Recently Choudhury and Kundu [5] extended Theorem 2.7 to Geraghty contractions in the context of coupled fixed points in metric spaces with partial order.

Theorem 2.11: [5] Let $(X, \le)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $\beta \in S$ such that

$$d(F(x,y), F(u,v)) \le \beta\left[\frac{d(x,u)+d(y,v)}{2}\right]\left(\frac{d(x,u)+d(y,v)}{2}\right) \qquad \ldots\ldots\ (2)$$

whenever $x, y, u, v \in X$ and (x, y) and (u, v) are comparable..

Also suppose that either (i) F is continuous or (ii) X has the following properties:

(a) If $\{x_n\}$ is a non-decreasing sequence in X with $x_n \to x$, then $x_n \le x\ \forall\, n \in \mathbb{Z}^+$

(b) If $\{y_n\}$ is a non- increasing sequence in X with $y_n \to y$, then $y_n \ge y\ \forall\, n \in \mathbb{Z}^+$

If $\exists\, x_0, y_0 \in X$ such that $x_0 \le F(x_0, y_0)$ and $y_0 \ge F(x_0, y_0)$, then F has a fixed point. That is $\exists\, x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

G.V.R. Babu and P. Subhashini [3] proved two theorems which extend the coupled fixed point results established by Gnana Bhaskar and Lakshmikantham [12] and Choudhury and Kundu [5], to the case of Geraghty contraction maps by using an altering distance function.

Theorem 2.12 (GVR Babu and P Subhashini, Theorem 2.1, [3]): Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a continuous map having the mixed monotone property on X. Suppose there exists an altering distance function φ and $\beta \in S$ such that

$$\varphi\big(d(F(x,y), F(u,v))\big) \leq \beta\left[\frac{d(x,u)+d(y,v)}{2}\right]\varphi(\max\{d(x,u), d(y,v)\}) \quad \ldots\ldots \ (3)$$

$\forall\, x, y, u, v \in X$ whenever (x, y) and (u, v) are comparable.
If $\exists\, x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(x_0, y_0)$, then F has a coupled fixed point. That is $\exists\, x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 2.13 (GVR Babu and P Subhashini, Theorem 2.2, [3]): Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a continuous map having the mixed monotone property on X. Suppose there exists an altering distance function φ and $\beta \in S$ such that

$$\varphi\big(d(F(x,y), F(u,v))\big) \leq \beta\left[\frac{d(x,u)+d(y,v)}{2}\right]\varphi(\max\{d(x,u), d(y,v)\}) \quad \ldots\ldots \ (4)$$

$\forall\, x, y, u, v \in X$ whenever (x, y) and (u, v) are comparable..
Further assume that X has the following properties:

(a) If $\{x_n\}$ is a non-decreasing sequence in X with $x_n \to x$, then $x_n \leq x\ \forall\, n \in \mathbb{Z}^+$
(b) If $\{y_n\}$ is a non- increasing sequence in X with $y_n \to y$, then $y_n \geq y\ \forall\, n \in \mathbb{Z}^+$

If $\exists\, x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point. That is, $\exists\, x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

In this paper, we establish Theorem 2.12 and Theorem 2.13 without using the continuity of alter distance function φ.

III. Main results

Theorem 3.1: Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X and there exist $\varphi \in \Phi$ and $\beta \in S$ such that

$$\varphi\big(d(F(x,y), F(u,v))\big) \leq \beta\left[\frac{d(x,u)+d(y,v)}{2}\right]\varphi(\max\{d(x,u), d(y,v)\}) \quad \ldots (5)$$

For all $x, y, u, v \in X$ whenever (x, y) and (u, v) are comparable..
Suppose $\exists\, x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$ for all $n = 0, 1, 2, \ldots$ (6)
Then $\{x_n\}$ is an increasing sequence, $\{y_n\}$ is a decreasing sequence and $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

Proof: First we prove that $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$ for all $n = 0, 1, 2, \ldots$ (7)
and then we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.
We have $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$.
Hence $x_0 \leq x_1$ and $y_0 \geq y_1$
$\therefore$ (7) is true for $n = 0$.
Assume that (7) is true for some positive integer n.
By using the mixed monotone property of F, we have
$x_{n+2} = F(x_{n+1}, y_{n+1}) \geq F(x_n, y_n) = x_{n+1}$ and $y_{n+2} = F(y_{n+1}, x_{n+1}) \leq F(y_n, x_n) = y_{n+1}$
$\therefore$ (7) is true for $n + 1$.
Therefore by mathematical induction (7) follows.
We now show that $\lim_{n\to\infty} \max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\} = 0$.
We have $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$ for all $n = 0, 1, 2, \ldots$

$$\text{Now } \varphi\big(d(x_{n+1}, x_n)\big) = \varphi\big(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))\big)$$
$$\leq \beta(t_n)\, \varphi(\max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}) \quad \ldots. \ (8)$$
$$\leq \varphi(\max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\})$$
$$\varphi\big(d(y_n, y_{n+1})\big) = \varphi\big(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))\big)$$
$$\leq \beta(t_n)\, \varphi(\max\{d(y_{n-1}, y_n), d(x_{n-1}, x_n)\}) \quad \ldots. \ (9)$$

where $t_n = \frac{d(y_{n-1},y_n)+d(x_{n-1},x_n)}{2}$

$\leq \varphi(\max\{d(y_{n-1},y_n), d(x_{n-1},x_n)\})$

From (8) and (9), we have

$\max\{\varphi\{d(x_n,x_{n+1}), d(y_n,y_{n+1})\}\} \leq \beta(t_n)\, \varphi(\max\{d(x_n,x_{n-1}), d(y_n,y_{n-1})\})$

Since φ is increasing, we get

$\varphi\{\max\{d(x_n,x_{n+1}), d(y_n,y_{n+1})\}\} \leq \beta(t_n)\, \varphi(\max\{d(x_n,x_{n-1}), d(y_n,y_{n-1})\})$ … (10)

If $\beta(t_n) = 0$, for some n, then $\varphi\{\max\{d((x_n,x_{n+1}), d((y_n,y_{n+1})\}\} = 0$ and hence from (10) we have $\varphi\{\max\{d(x_m,x_{m+1}), d(y_m,y_{m+1})\}\} = 0 \; for \;\; m \geq n$.

Consequently $\max\{d(x_m,x_{m+1}), d(y_m,y_{m+1})\} = 0$ for $m \geq n$

So that $\lim_{m\to\infty} \max\{d(x_m,x_{m+1}), d(y_m,y_{m+1})\} = 0$.

Hence, we may suppose that $\beta(t_n) > 0 \;\forall\, n$ ……. (11)

Again from non-decreasing property of φ and (10), we have

$\max\{d(x_n,x_{n+1}), d(y_n,y_{n+1})\} \leq \max\{d(x_n,x_{n-1}), d(y_n,y_{n-1})\}$

$\therefore$ $\max\{d(x_n,x_{n+1}), d(y_n,y_{n+1})\}$ is a non-negative and decreasing sequence of real and hence it converges to a real number r (say) $r \geq 0$.

Now, we prove that $r = 0$. If possible suppose that $r > 0$.

Again from (10), we have

$\varphi(a_{n+1}) \leq \beta(t_n)\varphi(a_n)$ where $a_n = \max\{d(x_n,x_{n-1}), d((y_n,y_{n-1})\} \geq r > 0$

i.e $\frac{\varphi(a_{n+1})}{\varphi(a_n)} \leq \beta(t_n) < 1 \;\; (\because \; \varphi(a_n) > \varphi(r) > 0)$

Let $\{\varphi(a_n)\}$ decreases to s, where $\lim_{n\to\infty} \varphi(a_n) = s$.

Now $r \leq a_{n+1} \Rightarrow \varphi(r) \leq \varphi(a_{n+1}) \;\forall\, n \;\Rightarrow\; \varphi(r) \leq s$

Now $\varphi(r) \leq s \leq \varphi(a_{n+1}) \leq \beta(t_n)\varphi(a_n)$ ….. (12)

Case (i): $\beta(t_n) \to 1$. Then $t_n \to 0 \;\; (\because \;\; \beta \in S)$

$\Rightarrow \frac{d(x_n,x_{n-1})+d(y_n,y_{n-1})}{2} \to 0$

$\Rightarrow d(x_n,x_{n-1}) + d(y_n,y_{n-1}) \to 0 \Rightarrow a_n \leq d(x_n,x_{n-1}) + d(y_n,y_{n-1}) \to 0 \;\Rightarrow a_n \to 0 \Rightarrow r = 0$

Case (ii): $\lim_{n\to\infty} \beta(t_n) \neq 1$.

Then $\exists\, \varepsilon > 0$ such that $\beta(t_n) < 1 - \varepsilon$ for infinitely many n.

Then from (10), we have $\varphi(r) \leq s \leq \varphi(a_{n+1}) \leq \beta(t_n)\varphi(a_n) \leq (1-\varepsilon)\varphi(a_n)$ for infinitely many n.

On letting $n \to \infty$, we get $\varphi(r) \leq s \leq (1-\varepsilon)s \;\Rightarrow s = 0$ and $\varphi(r) = 0 \Rightarrow s = 0$ and $r = 0$.

$\therefore 0 = r = \lim_{n\to\infty}(\max\{d((x_n,x_{n+1}), d((y_n,y_{n+1})\})$ …. (13)

Next, we have to prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

If possible, assume that either $\{x_n\}$ or $\{y_n\}$ fails to be Cauchy.

Then either $\lim_{m,n\to\infty} d(x_m,x_n) \neq 0$ or $\lim_{m,n\to\infty} d(y_m,y_n) \neq 0$

Hence $\max\{\lim_{m,n\to\infty} d(x_m,x_n), \lim_{m,n\to\infty} d(y_m,y_n)\} \neq 0$

i.e. $\lim_{m,n\to\infty} \max\{\lim_{m,n\to\infty} d(x_m,x_n), \lim_{m,n\to\infty} d(y_m,y_n)\} \neq 0$

i.e $\exists\, \varepsilon > 0$, for which we can find sub sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n(k) > m(k) > k$ such that $\max\{d(x_{m(k)},x_{n(k)}), d(y_{m(k)},y_{n(k)})\} \geq \varepsilon$ ….. (14)

Further, we choose $n(k)$ to be the smallest +ve integer such that $n(k) > m(k)$ satisfying (14).

Hence, we have $\max\{d(x_{m(k)},x_{n(k)}), d(y_{m(k)},y_{n(k)})\} \geq \varepsilon$ and

$\max\{d(x_{m(k)},x_{n(k)-1}), d(y_{m(k)},y_{n(k)-1})\} < \varepsilon$.…… (15)

Now, we prove that

I: $\lim_{k\to\infty} \max\{d(x_{n(k)},x_{m(k)}), d(y_{n(k)},y_{m(k)})\} = \varepsilon$

II: $\lim_{k\to\infty} \max\{d(x_{n(k)-1},x_{m(k)-1}), d(y_{n(k)-1},y_{m(k)-1})\} = \varepsilon$

III: $\lim_{k\to\infty} \max\{d(x_{m(k)},x_{n(k)-1}), d(y_{m(k)},y_{n(k)-1})\} = \varepsilon$

First we prove I:-

From the triangular inequality and (11), we have

$d(x_{n(k)},x_{m(k)}) \leq d(x_{n(k)},x_{n(k)-1}) + d(x_{n(k)-1},x_{m(k)}) < d(x_{n(k)},x_{n(k)-1}) + \varepsilon$ … (16)

$d(y_{n(k)},y_{m(k)}) \leq d(y_{n(k)},y_{n(k)-1}) + d(y_{n(k)-1},y_{m(k)}) < d(y_{n(k)},y_{n(k)-1}) + \varepsilon$ … (17)

From (14), (16) and (17)

$\varepsilon \leq \max\{d(x_{n(k)},x_{m(k)}), d(y_{n(k)},y_{m(k)})\} < \max\{d(x_{n(k)},x_{n(k)-1}), d(y_{n(k)},y_{n(k)-1})\}$ … (18)

On letting $k \to \infty$, we get

$\varepsilon \leq \lim_{k\to\infty} \max\{d(x_{n(k)},x_{m(k)}), d((y_{n(k)},y_{m(k)})\} \leq \varepsilon$

$\therefore \lim_{k\to\infty} \max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\} = \varepsilon$

$\therefore$ (I) holds.

Now, we prove (II):-

$$d(x_{n(k)-1}, x_{m(k)-1}) \le d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)-1}) < d(x_{m(k)-1}, x_{m(k)}) + \varepsilon \;\; (by\ (15))$$

$$d(y_{n(k)-1}, y_{m(k)-1}) \le d(y_{m(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)-1}) < d(y_{m(k)-1}, y_{m(k)}) + \varepsilon \;\; (by\ (15))$$

$\therefore \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} \le \max\{d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})\} + \varepsilon$

$\therefore \limsup \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} \le \varepsilon$ … (19)

$$d(x_{n(k)}, x_{m(k)}) \le d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$

$$d(y_{n(k)}, y_{m(k)}) \le d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})$$

$\therefore \max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\} \le \max\{d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)-1}, y_{n(k)})\} + \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} + \max\{d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})\}$

Letting $k \to \infty$, from (12), we get

$$0 \le \liminf \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} \le \limsup \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} \le \varepsilon$$

$\therefore \lim \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} = \varepsilon$

$\therefore$ (II) holds.

$$d(x_{n(k)}, x_{m(k)}) \le d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \le d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$

$$d(y_{n(k)}, y_{m(k)}) \le d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \le d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})$$

$\therefore \max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\} \le \max\{d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{m(k)})\} + \max\{d(x_{n(k)-1}, x_{m(k)}), d(y_{n(k)-1}, y_{m(k)})\}$

$\le \max\{d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})\} + \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} + \max\{d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})\}$

On letting $k \to \infty$, from (I), (II), we get (III). $\left(\because\ \varepsilon \le 0 + \lim \max\{d(x_{n(k)-1}, x_{m(k)}), d(y_{n(k)-1}, y_{m(k)})\} \le \varepsilon\right)$

Now, we have $\varepsilon \le \liminf \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}$
$\le \limsup \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} = \varepsilon$

$\therefore \liminf \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} = \varepsilon$

Since $x_{n(k)-1} \ge x_{m(k)-1}$ and $y_{n(k)-1} \le y_{m(k)-1}$, from (5), we get

$$\varphi\left(d(x_{n(k)}, x_{m(k)})\right) = \varphi\left(d\left(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1})\right)\right)$$

$$\le \beta\left(\frac{d(x_{n(k)-1}, x_{m(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1})}{2}\right) \varphi\left(\max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}\right) \quad \ldots\ldots. \ (20)$$

Similarly $\varphi\left(d(y_{n(k)}, y_{m(k)})\right) = \varphi\left(d\left(F(y_{n(k)-1}, x_{m(k)-1}), F(y_{n(k)-1}, x_{n(k)-1})\right)\right)$

$$\le \beta\left(\frac{d(x_{n(k)-1}, x_{m(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1})}{2}\right) \varphi\left(\max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}\right) \quad \ldots\ldots.. \ (21)$$

From (20) and (21), we get $\varphi(s_k) \le \beta(q_k).\varphi(p_k)$ where

$s_k = \max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\}$ and $q_k = \frac{d(x_{n(k)-1}, x_{m(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1})}{2}$

$p_k = \max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}$

Now, $\beta(q_k) \to 1 \Rightarrow q_k \to 0$ (by hypothesis)
$\Rightarrow p_k \to 0\ as\ k \to \infty \Rightarrow \varepsilon = 0$ by (16) , a contradiction.

$\therefore \lim_{k\to\infty} \beta(q_k) \ne 1$

$\therefore \exists$ a positive integer N and $\delta \in (0,1)$ such that $\beta(q_k) < \delta$ for $k \ge N$ …. (22)

$\therefore \varphi(\varepsilon) < \varphi(s_k) \le \beta(q_k)\,\varphi(p_k)$ ….. (23)

From (22), we have $\varphi(\varepsilon) < \varphi(s_k) < \delta\varphi(p_k) < \delta\varphi(\varepsilon + \eta)$ for a given $\eta > 0$ and large k.
$\varphi(\varepsilon) \le \varphi(\varepsilon + 0) < \varphi(s_k) < \delta\varphi(\varepsilon + \eta)$
This being true for every $\eta > 0$ follows that $0 \le \varphi(\varepsilon) < \varphi(\varepsilon + 0) < \delta\varphi(\varepsilon + 0)$
$\therefore 0 \le \varphi(\varepsilon) < \varphi(\varepsilon + 0) = 0$ ($\because 0 < \delta < 1$)
$\therefore\ \varphi(\varepsilon) = 0 \Rightarrow \varepsilon = 0$, a contradiction.
Hence both $\{x_n\}$ *and* $\{y_n\}$ are Cauchy sequences.

Theorem 3.2: In addition to the hypothesis of Theorem 3.1, suppose that (X, d) is complete and either (a) F is continuous or (ii) X has the following properties:

(a) If $\{x_n\}$ is a non-decreasing sequence in X with $x_n \to x$, then $x_n \le x\ \forall\, n \in \mathbb{Z}^+$
(b) If $\{y_n\}$ is a non- increasing sequence in X with $y_n \to y$, then $y_n \ge y\ \forall\, n \in \mathbb{Z}^+$

Then F has a coupled fixed point in X. i.e. $\exists\, x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof: Let the sequences $\{x_n\}$ and $\{y_n\}$ be as defined in (6) of Theorem 3.1. Then from Theorem 3.1, both the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy. Since (X, d) is complete, $\exists\, x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$.

(a) Suppose F is continuous. Then it follows that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = F\left(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}\right) = F(x, y)$$
$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}) = F(y, x)$$

Thus (x, y) is a coupled fixed point of F in X.

(b) Suppose that X has the properties (i) and (ii). Let $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y, \exists$ a positive integer N such that $d(x, x_n) < \varepsilon$ and $d(y, y_n) < \varepsilon\ \ \forall\, n \ge N$.

Since $x_n \le x$ and $y_n \ge y,\ \forall\, n$, from (5), we have

$$\varphi\big(d(F(x, y), F(x_n, y_n))\big) \le \beta\left[\frac{d(x, x_n) + d(y, y_n)}{2}\right]\varphi(\max\{d(x, x_n), d(y, y_n)\})$$
$$\begin{cases} = 0\ \ if\ \ \beta\left(\frac{d(x,x_n)+d(y,y_n)}{2}\right) = 0\ \ or\ \ \varphi(\max\{d(x, x_n), d(y, y_n)\}) = 0 \\ <\ \varphi(\max\{d(x, x_n), d(y, y_n)\}) \quad otherwise \\ \le \varphi(\varepsilon) \quad for\ all\ \ n \ge N \end{cases}$$

$\therefore\ d(F(x, y), F(x_n, y_n)) < \varepsilon$ for all $n \ge N$ ($\because\ \varphi$ is increasing)
$\therefore\ F(x_n, y_n) \to F(x, y)$. But $F(x_n, y_n) = x_{n+1} \to x\ as\ n \to \infty$
$\therefore x = F(x, y)$ ….. (24)
Again from (5), we have

$$\varphi\Big(d\big(F(y, x), F(y_n, x_n)\big)\Big) = \varphi\big(d(F(y_n, x_n), F(y, x))\big)$$
$$\le \beta\left[\frac{d(y_n, y) + d(x_n, x)}{2}\right]\varphi(\max\{d(y_n, y), d(x_n, x)\})$$
$$\begin{cases} = 0\ \ if\ \ \beta\left(\frac{d(y_n,y)+d(x_n,x)}{2}\right) = 0\ \ or\ \ \varphi(\max\{d(y_n, y), d(x_n, x)\}) = 0 \\ <\ \varphi(\max\{d(y_n, y), d(x_n, x)\}) \quad otherwise \\ < \varphi(\varepsilon) \quad for\ all\ \ n \ge N \end{cases}$$

$\therefore\ d\big(F(y, x), F(y_n, x_n)\big) < \varepsilon\ \forall\, n \ge N$ ($\because\ \varphi$ is increasing)
$\therefore\ F(y_n, x_n) \to F(x, y)$. But $F(y_n, x_n) = y_{n+1} \to y\ \ as\ n \to \infty$
$\therefore y = F(y, x)$ ….. (25)
$\therefore$ (24) and (25) shows that (x, y) is a coupled fixed point for F in X.

Lemma 3.3: Let $(X, \le)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose there exists an altering distance function $\varphi \in \Phi$ and $\beta \in S$ such that

$$\varphi\big(d(F(x, y), F(u, v))\big) \le \beta\left[\tfrac{d(x,u)+d(y,v)}{2}\right]\varphi(\max\{d(x, u), d(y, v)\}) \quad \dots (5)$$

for $x, y, u, v \in X$ whenever (x, y) and (u, v) are comparable.
Suppose (x, y) is a coupled fixed point of F. i.e. $x = F(x, y)$ or $y = F(y, x)$. Let $(u, v) \in X \times X$ such that $(u, v) \le_1 (x, y)$ … (26)
Construct the sequences $\{u_n\}$ and $\{v_n\}$ by $u_0 = u, v_0 = v, u_{n+1} = F(u_n, v_n), v_{n+1} = F(v_n, u_n)$.
Then $u_n \to x$ and $v_n \to y$ as $n \to \infty$.

Proof: First we prove that $(u_n, v_n) \leq_1 (x, y)$ for all $n = 0, 1, 2, ...$
i.e $x \geq u_n$ and $y \geq v_n$ for all $n = 1, 2, ...$ (27)
From (26), $x \geq u = u_0$ and $y \leq v = v_o$
$\therefore$ (27) is true for $n = 0$.
Assume that (27) is true for some positive integer n.
Hence $u_{n+1} = F(u_n, v_n) \leq F(x, y) = x$ and $v_{n+1} = F(v_n, u_n) \geq F(y, x) = y$
Therefore by mathematical induction (27) is true for all $n = 1, 2, ...$
Since $x \geq u_{n-1}$ and $y \leq v_{n-1}$ and from (5), we have,
$\varphi\big(d(x, u_n)\big) = \varphi(d(F(x, y), F(u_{n-1}, v_{n-1}) \quad \leq \beta\left[\frac{d(x,u_{n-1})+d(y,v_{n-1})}{2}\right]\varphi(\max\{d(x, u_{n-1}), d(y, v_{n-1})\})$
and $\varphi\big(d(y, v_n)\big) = \varphi\big((d(F(y, x), F(v_{n-1}, u_{n-1}))\big)$
$\leq \beta\left[\frac{d(y,v_{n-1})+d(x,u_{n-1})}{2}\right]\varphi(\max\{d(y, v_{n-1}) + d(x, u_{n-1})\})$
$\therefore \max\{\varphi\big(d(x, u_n)\big), \varphi\big(d(y, v_n)\big)\} \leq$
$\beta\left[\frac{d(y,v_{n-1})+d(x,u_{n-1})}{2}\right]\varphi(\max\{d(y, v_{n-1}), d(x, u_{n-1})\})$ …. (28)
Case (i): $\varphi(\max\{d(x, u_{N-1}), d(y, v_{N-1})\}) = 0$ for some N
Then $x = u_{N-1}$ and $y = v_{N-1}$ and from (28), we get $x = u_N$ and $y = v_N$.
Similarly $x = u_n$ or $y = v_n$ for $n \geq N$.
$\therefore$ $u_n \to x$ and $v_n \to y$ as $n \to \infty$.
Case (ii): $\varphi(\max\{d(x, u_{n-1}), d(y, v_{n-1})\}) = 0$ for all $n \geq 1$.
Then from (28), we have $\max\{\varphi\big(d(x, u_n)\big)\varphi\big(d(y, v_n)\big)\} \leq \varphi(max\ \{d(x, u_{n-1}), d(y, v_{n-1})\})$
Since φ is increasing, we get $\max\{d(x, u_n), d(y, v_n)\} \leq \max\big(d(x, u_{n-1}), d(y, v_{n-1})\big)$
i.e. $\max\{d(x, u_n), d(y, v_n)\}$ is a deceasing sequence of reals and hence deceases to r (say), $r \geq 0$.
i.e. $\lim_{n\to\infty} \max\{d(x, u_n), d(y, v_n)\} = r$.
Again from (28), we have
$\varphi(r) \leq \varphi(\max\{d(x, u_n), d(y, v_n)\}) \leq$
$\beta\left(\frac{d(x,u_{n-1})+d(y,v_{n-1})}{2}\right)\varphi(\max\{d(x, u_{n-1}), d(y, v_{n-1})\})$ … (29)
Let $s_n = \max\{d(x, u_n), d(y, v_n)\}$, $q_n = \frac{d(x,u_{n-1})+d(y,v_{n-1})}{2}$
Then $\max\{d(x, u_{n-1}), d(y, v_{n-1})\} = s_{n-1}$
$\therefore$ $\varphi(r) \leq \varphi(s_n) \leq \beta(q_n)\ \varphi(s_{n-1})$
Suppose $r > 0$. Now, $\beta(q_k) \to 1 \Rightarrow q_n \to 0$ (by hypothesis)
$\Rightarrow s_n \to 0\ as\ n \to \infty \Rightarrow r = 0$, ($\because$ $s_{n-1} \to s$), a contradiction.
$\therefore$ $\lim_{k\to\infty} \beta(q_n) \neq 1$
$\therefore$ $\exists$ a positive integer N and $\delta \in (0,1)$ such that $\beta(q_n) < \delta$ for $n \geq N$
$\therefore$ $0 < \varphi(r) \leq \varphi(s_n) < \delta\varphi(s_{n-1}) \leq \delta\varphi(r + \eta)$ for a given $\eta > 0$ and large n.
$\therefore$ $\varphi(r) \leq \varphi(r + 0) \leq \varphi(s_n) < \delta\varphi(r + \eta)$
$\therefore$ $0 < \varphi(r) \leq \varphi(r + 0) \leq \delta\varphi(r + 0) < \varphi(r + 0)$ ($\because$ $0 < \delta < 1$), a contradiction.
$\therefore r = 0$. Therefore $d(x, u_n) \to 0$ and $d(y, v_n) \to 0$ as $n \to \infty$.
$\therefore$ $u_n \to x$ and $v_n \to y$ as $n \to \infty$.

Lemma 3.4: Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose there exists an altering distance function $\varphi \in \Phi$ and $\beta \in S$ such that
$\varphi\big(d(F(x, y), F(u, v))\big) \leq \beta\left[\frac{d(x,u)+d(y,v)}{2}\right]\varphi(\max\{d(x, u), d(y, v)\})$ … (5) for $x, y, u, v \in X$ whenever (x, y) and (u, v) are comparable.
Suppose (x, y) is a coupled fixed point of F. i.e. $x = F(x, y)$ or $y = F(y, x)$. Let $(u, v) \in X \times X$ such that $(x, y) \leq_1 (u, v)$ … (30)
Construct the sequences $\{u_n\}$ and $\{v_n\}$ by $u_0 = u, v_0 = v, u_{n+1} = F(u_n, v_n), v_{n+1} = F(v_n, u_n)$.
Then $u_n \to x$ and $v_n \to y$ as $n \to \infty$.
The proof is similar to that of Lemma 3.3.

Definition 3.5: Let $(u, v) \in X \times X$. Define the sequences $\{x_n\}\ and\ \{y_n\}$ as follows:
$u_0 = u, v_0 = v, u_1 = F(u_0, v_0)$ and $u_{n+1} = F(u_n, v_n)$ for all $n \geq 1$
$v_1 = F(v_0, u_0)$ and $v_{n+1} = F(v_n, u_n)$ for all $n \geq 1$

Then the sequence $\{(u_n, v_n)\}$ is called the coupled iterative sequence of (u, v).

Theorem 3.6: Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a metric space. Let $F: X \times X \to X$ be a continuous map having the mixed monotone property on X. Suppose there exists an altering distance function $\varphi \in \Phi$ and $\beta \in S$ such that
$\varphi\big(d(F(x,y), F(u,v))\big) \leq \beta\left[\frac{d(x,u)+d(y,v)}{2}\right]\varphi(\max\{d(x,u), d(y,v)\})$ … (5)
for $x, y, u, v \in X$ whenever (x, y) and (u, v) are comparable.
Suppose (x, y) is a coupled fixed point of F. i.e. $x = F(x, y)$ or $y = F(y, x)$. Let $(u, v) \in X \times X$ such that (u, v) *and* (x, y) are comparable. Let $\{(u_n, v_n)\}$ be the coupled iterative sequence of (u, v).
Then $u_n \to x$ *and* $v_n \to y$.

Proof: Case(i): If $(u, v) \leq (x, y)$, then result follows from Lemma 3.3.
Case(ii): If $x, y) \leq (u, v)$, then the result follows from Lemma 3.4.

Theorem 3.7: Suppose the hypothesis of Theorem 3.2 holds. Then there does not exists a pair $(u, v) \in X \times X$ such that (u, v) is comparable to two distinct coupled fixed points of F.

Proof: Suppose (x, y) and (x', y') are two coupled fixed points of F. Let (u, v) be comparable with (x, y) and (x', y'). Let $\{(u_n, v_n)\}$ be the coupled iterative sequence of (u, v).
Case(i): Suppose $(u, v) \leq_1 (x, y)$ and $(u, v) \leq_1 (x', y')$.
Then by Lemma 3.3, $u_n \to x$ *and* $v_n \to y$ and also $u_n \to x'$ *and* $v_n \to y'$.
$\therefore x = x'$ and $y = y'$. Consequently $(x, y) = (x', y')$.
Case(ii): Suppose $(x, y) \leq_1 (u, v)$ and $(x', y') \leq_1 (u, v)$.
Then by Lemma 3.4, $u_n \to x$ *and* $v_n \to y$ and also $u_n \to x'$ *and* $v_n \to y'$.
$\therefore x = x'$ and $y = y'$. Consequently $(x, y) = (x', y')$.
Case(iii): Suppose $(x, y) \leq_1 (u, v) \leq_1 (x', y')$.
Then by Lemma 3.4, $u_n \to x$ *and* $v_n \to y$ and also $u_n \to x'$ *and* $v_n \to y'$.
$\therefore x = x'$ and $y = y'$. Consequently $(x, y) = (x', y')$.
Case(iv): Suppose $(x', y') \leq_1 (u, v) \leq_1 (x, y)$.
In this case also $(x, y) = (x', y')$ as in Case (iii).
Hence (x, y) and (x', y') cannot be distinct, a contradiction.

Theorem 3.8: Suppose the hypothesis of Theorem 3.2 holds. Further assume that
(H): For $(x, y), (z, t) \in X \times X$ there exists $(u, v) \in X \times X$ which is comparable to (x, y) and (z, t). Then F has unique coupled fixed point.

Proof: By Theorem 3.2, F has a coupled fixed point (x, y).
Suppose (x', y') is also a coupled fixed point of F.
Then by (H) there exists $(u, v) \in X \times X$ which is comparable to (x, y) and (x', y').
Then by Theorem 3.7, $(x, y) = (x', y')$.
Thus F has unique coupled fixed point.

Corollary 3.9: Suppose the hypothesis of Theorem 3.2 holds. Further assume that $(X, \leq)$ is a lattice. Then F has a unique coupled fixed point.

Proof: Since $(X, \leq)$ is a lattice, condition (H) holds.
Consequently by Theorem 3.8, F has a unique coupled fixed point.

Note: If we replace the argument $\frac{d(x,u)+d(y,v)}{2}$ in β by $\max\{d(x,u), d(y,v)\}$ still the results hold good.

Conclusion: Under the hypothesis of Theorem 3.2, the fixed point set $\mathfrak{F}$ of F decomposes the set X into pointwise disjoint sets $\{S_p \,/ p \in \mathfrak{F}\}$ in the following way:
For $p \in \mathfrak{F}$, write $S_p = \{a \in X: p \textit{ is comparable with } a\}$
Then (i) $S_p \neq \emptyset$, since $p \in S_p$
(ii) S_p and S_q are disjoint whenever $p, q \in \mathfrak{F}$ and $p \neq q$

(iii) $S = \bigcup_{p\in\mathfrak{F}} S_p$ may be a proper sub set of X
in which case $X - S$ does not contain any fixed point of F.

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